On automorphism groups of fiber bundles

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Abstract

We obtain analogues of classical results on automorphism groups of holomorphic fiber bundles, in the setting of group schemes. Also, we establish a lifting property of the connected automorphism group, for torsors under abelian varieties. These results will be applied to the study of homogeneous bundles over abelian varieties.

1 Introduction

This work arose from a study of homogeneous bundles over an abelian variety A, that is, of those principal bundles with base A and fiber an algebraic group G, that are isomorphic to all of their pull-backs by the translations of A (see [Br2]). In the process of that study, it became necessary to obtain algebro-geometric analogues of two classical results about automorphisms of fiber bundles in complex geometry. The first one, due to Morimoto (see [Mo]), asserts that the equivariant automorphism group of a principal bundle over a compact complex manifold, with fiber a complex Lie group, is a complex Lie group as well. The second one, a result of Blanchard (see [Bl]), states that a holomorphic action of a complex connected Lie group on the total space of a locally trivial fiber bundle of complex manifolds descends to a holomorphic action on the base, provided that the fiber is compact and connected.

Also, we needed to show the existence in the category of schemes of certain fiber bundles associated to a G-torsor (or principal bundles) $\pi : X \to Y$, where G is a connected group scheme and X, Y are algebraic schemes; namely, those fiber bundles $X \times^G Z \to Y$ associated to G-homogeneous varieties Z. Note that the fiber bundle associated to an arbitrary G-scheme Z exists in the category of algebraic spaces, but may fail to be a scheme (see [Bi, KM]).

Finally, we were led to a lifting result which reduces the study of homogeneous bundles to the case that the structure group is linear, and does not seem to have its holomorphic

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counterpart. It asserts that given a G-torsor $\pi : X \to Y$ where G is an abelian variety and X, Y are smooth complete algebraic varieties, the connected automorphism group of X maps onto that of Y under the homomorphism provided by the analogue of Blanchard's theorem.

In this paper, we present these preliminary results which may have independent interest, with (hopefully) modest prerequisites. Section 2 is devoted to a scheme-theoretic version of Blanchard's theorem: a proper morphism of schemes $\pi : X \to Y$ such that $\pi_*(\mathcal{O}_X) = \mathcal{O}_Y$ induces a homomorphism $\pi_* : \mathrm{Aut}^o(X) \to \mathrm{Aut}^o(Y)$ between the neutral components of the automorphism group schemes (Corollary 2.2). Our proof is an adaptation of that given in [Ak] in the setting of complex spaces.

In Section 3, we consider a torsor $\pi : X \to Y$ under a connected group scheme G, and show the existence of the associated fiber bundle $X \times^G G/H = X/H$ for any subgroup scheme $H \subset G$ (Theorem 3.3). As a consequence, $X \times^G Z$ exists when Z is the total space of a G-torsor, or a group scheme where G acts via a homomorphism (Corollary 3.4). Another application of Theorem 3.3 concerns the quasi-projectivity of torsors (Corollary 3.5); it builds on work of Raynaud, who showed e.g. the local quasi-projectivity of homogeneous spaces over a normal scheme (see [Ra]).

The automorphism groups of torsors are studied in Section 4. In particular, we obtain a version of Morimoto's theorem: the equivariant automorphisms of a torsor over a proper scheme form a group scheme, locally of finite type (Theorem 4.2). Here our proof, based on an equivariant completion of the structure group, is quite different from the original one. We also analyze the relative equivariant automorphism group of such a torsor; this yields a version of Chevalley's structure theorem for algebraic groups in that setting (Proposition 4.3).

The final Section 5 contains a full description of relative equivariant automorphisms for torsors under abelian varieties (Proposition 5.1) and our lifting result for automorphisms of the base (Theorem 5.4).

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Notation and conventions. Throughout this article, we consider algebraic varieties, schemes, and morphisms over an algebraically closed field k. Unless explicitly mentioned, we will assume that the considered schemes are of finite type over k (such schemes are also called algebraic schemes). By a point of a scheme X, we will mean a closed point unless explicitly mentioned. A *variety* is an integral separated scheme.

We will use [DG] as a general reference for group schemes. Given such a group scheme G, we denote by $\mu_G : G \times G \to G$ the multiplication and by $e_G \in G(k)$ the neutral element. The neutral component of G is denoted by G^o, and the Lie algebra by $\mathrm{Lie}(G)$.

We recall that an *action* of G on a scheme X is a morphism

$$\alpha : G \times X \longrightarrow X, \quad (g, x) \longmapsto g \cdot x$$

such that the composite map

$$X \xrightarrow{e_G \times \mathrm{id}_X} G \times X \xrightarrow{\alpha} X$$

is the identity, and the square

$$\begin{array}{ccc} G \times G \times X & \xrightarrow{\mathrm{id}_G \times \alpha} & G \times X \\ {\scriptstyle \mu_G \times \mathrm{id}_X} \big\downarrow & & \big\downarrow {\scriptstyle \alpha} \\ G \times X & \xrightarrow{\alpha} & X \end{array}$$

commutes. We then say that X is a *G-scheme*. A morphism $f : X \to Y$ between two G-schemes is called *equivariant* if the square

$$\begin{array}{ccc} G \times X & \xrightarrow{\alpha} & X \\ {\scriptstyle \mathrm{id}_G \times f} \big\downarrow & & \big\downarrow {\scriptstyle f} \\ G \times Y & \xrightarrow{\beta} & Y \end{array}$$

commutes (with the obvious notation). We then say that f is a *G-morphism*.

A smooth group scheme will be called an algebraic group. By Chevalley's structure theorem (see [Ro, Theorem 16], or [Co] for a modern proof), every connected algebraic group G has a largest closed connected normal affine subgroup G_{aff}; moreover, the quotient $G/G_{\mathrm{aff}} =: A(G)$ is an abelian variety. This yields an exact sequence of connected algebraic groups

$$1 \longrightarrow G_{\mathrm{aff}} \longrightarrow G \longrightarrow A(G) \longrightarrow 1.$$

2 Descending automorphisms for fiber spaces

We begin with the following scheme-theoretic version of a result of Blanchard (see [Bl, Section I.1] and also [Ak, Lemma 2.4.2]).

PROPOSITION 2.1. *Let G be a connected group scheme, X a G-scheme, Y a scheme, and $\pi : X \to Y$ a proper morphism such that $\pi_*(\mathcal{O}_X) = \mathcal{O}_Y$. Then there is a unique G-action on Y such that π is equivariant.*

Proof. We will consider a scheme Z as the ringed space $(Z(k), \mathcal{O}_Z)$ where the set $Z(k)$ is equipped with the Zariski topology; this makes sense as Z is of finite type.

We first claim that the abstract group $G(k)$ permutes the fibers of $\pi : X(k) \to Y(k)$ (note that these fibers are non-empty and connected, since $\pi_*(\mathcal{O}_X) = \mathcal{O}_Y$). Let $y \in Y(k)$ and denote by F_y the set-theoretic fiber of π at y, viewed as a closed reduced subscheme of X. Then the map

$$\varphi : G_{\mathrm{red}} \times F_y \longrightarrow Y, \quad (g, x) \longmapsto \pi(g \cdot x)$$

maps $\{e_G\} \times F_y$ to the point y. Moreover, G_{red} is a variety, and F_y is connected and proper. By the rigidity lemma (see [Mu, p. 43]), it follows that φ maps $\{g\} \times F_y$ to a point for any $g \in G(k)$, i.e., $g \cdot F_y \subset F_{g \cdot y}$. Thus, $g^{-1} \cdot F_{g \cdot y} \subset F_y$ and hence $g \cdot F_y = F_{g \cdot y}$. This implies our claim.

That claim yields a commutative square

$$\begin{array}{ccc} G(k) \times X(k) & \xrightarrow{\alpha} & X(k) \\ {\scriptstyle \mathrm{id}_G \times \pi} \downarrow & & \downarrow {\scriptstyle \pi} \\ G(k) \times Y(k) & \xrightarrow{\beta} & Y(k), \end{array}$$

where β is an action of the (abstract) group $G(k)$.

Next, we show that β is continuous. It suffices to show that $\beta^{-1}(Z)$ is closed for any closed subset $Z \subset Y(k)$. But $(\mathrm{id}_G, \pi)^{-1} \beta^{-1}(Z) = \alpha^{-1} \pi^{-1}(Z)$ is closed, and (id_G, π) is proper and surjective; this yields our assertion.

Finally, we define a morphism of sheaves of k-algebras

$$\beta^\# : \mathcal{O}_Y \longrightarrow \beta_*(\mathcal{O}_{G \times Y}).$$

For this, to any open subset $V \subset Y$, we associate a homomorphism of algebras

$$\beta^\#(V) : \mathcal{O}_Y(V) \longrightarrow \mathcal{O}_{G \times Y}(\beta^{-1}(V)).$$

By assumption, the left-hand side is isomorphic to $\mathcal{O}_X(\pi^{-1}(V))$, and the right-hand side to

$$\mathcal{O}_{G \times X}((\mathrm{id}_G, \pi)^{-1} \beta^{-1}(V)) = \mathcal{O}_{G \times X}(\alpha^{-1} \pi^{-1}(V)).$$

We define $\beta^\#(V) := \alpha^\#(\pi^{-1}(V))$. Now it is straightforward to verify that $(\beta, \beta^\#)$ is a morphism of locally ringed spaces; this yields a morphism of schemes $\beta : G \times Y \to Y$. By construction, β is the unique morphism such that the square

$$\begin{array}{ccc} G \times X & \xrightarrow{\alpha} & X \\ {\scriptstyle \mathrm{id}_G \times \pi} \downarrow & & \downarrow {\scriptstyle \pi} \\ G \times Y & \xrightarrow{\beta} & Y \end{array}$$

commutes.

It remains to show that β is an action of the group scheme G. Note that e_G acts on $X(k)$ via the identity; moreover, the composite morphism of sheaves

$$\mathcal{O}_Y \xrightarrow{\beta^\#} \beta_*(\mathcal{O}_{G\times Y}) \xrightarrow{(e_G\times \mathrm{id}_Y)^\#} \beta_*(\mathcal{O}_{\{e_G\}\times Y}) \cong \mathcal{O}_Y$$

is the identity, since so is the analogous morphism

$$\mathcal{O}_X \xrightarrow{\alpha^\#} \alpha_*(\mathcal{O}_{G\times X}) \xrightarrow{(e_G\times \mathrm{id}_X)^\#} \alpha_*(\mathcal{O}_{\{e_G\}\times X}) \cong \mathcal{O}_X$$

and $\pi_*(\mathcal{O}_X) = \mathcal{O}_Y$. Likewise, the square

$$\begin{array}{ccc} G\times G\times Y & \xrightarrow{\mathrm{id}_G\times\beta} & G\times Y \\ {\scriptstyle \mu_G\times \mathrm{id}_Y}\downarrow & & {\scriptstyle\beta}\downarrow \\ G\times Y & \xrightarrow{\beta} & Y \end{array}$$

commutes on closed points, and the corresponding square of morphisms of sheaves commutes as well, since the analogous square with Y replaced by X commutes. $\square$

This proposition will imply a result of descent for group scheme actions, analogous to [Bl, Proposition I.1] (see also [Ak, Proposition 2.4.1]). To state that result, we need some recollections on automorphism functors.

Given a scheme S, we denote by $\mathrm{Aut}_S(X\times S)$ the group of automorphisms of $X\times S$ viewed as a scheme over S. The assignement $S\mapsto \mathrm{Aut}_S(X\times S)$ yields a group functor $Aut(X)$, i.e., a contravariant functor from the category of schemes to that of groups. If X is proper, then $Aut(X)$ is represented by a group scheme $\mathrm{Aut}(X)$, locally of finite type (see [MO, Theorem 3.7]). In particular, the neutral component $\mathrm{Aut}^o(X)$ is a group scheme of finite type. Also, recall that

$$\mathrm{Lie}\,\mathrm{Aut}(X) \cong \Gamma(X, T_X) \tag{1}$$

where the right-hand side denotes the Lie algebra of global vector fields on X, that is, of derivations of $\mathcal{O}_X$.

We now are in a position to state:

Corollary 2.2. *Let $\pi : X\to Y$ be a morphism of proper schemes such that $\pi_*(\mathcal{O}_X) = \mathcal{O}_Y$. Then π induces a homomorphism of group schemes*

$$\pi_* : \mathrm{Aut}^o(X) \longrightarrow \mathrm{Aut}^o(Y).$$

PROOF. This is a formal consequence of Proposition 2.1. Specifically, let $G := \mathrm{Aut}^o(X)$ and consider the G-action on Y obtained in that proposition. This yields a automorphism of $Y \times G$ as a scheme over G,

$$(y, g) \longmapsto (g \cdot y, g),$$

and in turn a morphism (of schemes)

$$\pi_* : G \longrightarrow \mathrm{Aut}(Y).$$

Moreover, $\pi_*(e_G) = e_{\mathrm{Aut}(Y)}$ since e_G acts via the identity. As G is connected, it follows that the image of π_* is contained in $\mathrm{Aut}^o(Y) =: H$. In other words, we have a morphism of schemes $\pi_* : G \to H$ such that $\pi_*(e_G) = e_H$. It remains to check that π_* is a homomorphism; but this follows from the fact that π_* corresponds to the G-action on Y, and hence yields a morphism of group functors. □

Given two complete varieties X and Y, the preceding corollary applies to the projections

$$p : X \times Y \to X, \quad q : X \times Y \to Y$$

and yields homomorphisms

$$p_* : \mathrm{Aut}^o(X) \times \mathrm{Aut}^o(Y) \to \mathrm{Aut}^o(X), \quad q_* : \mathrm{Aut}^o(X) \times \mathrm{Aut}^o(Y) \to \mathrm{Aut}^o(Y).$$

This implies readily the following analogue of [Bl, Corollaire, p. 161]:

COROLLARY 2.3. *Let X and Y be complete varieties. Then the homomorphism*

$$(p_*, q_*) : \mathrm{Aut}^o(X \times Y) \longrightarrow \mathrm{Aut}^o(X) \times \mathrm{Aut}^o(Y)$$

is an isomorphism, with inverse the natural homomorphism

$$\mathrm{Aut}^o(X) \times \mathrm{Aut}^o(Y) \longrightarrow \mathrm{Aut}^o(X \times Y), \quad (g, h) \longmapsto ((x, y) \mapsto (g(x), h(y)).$$

More generally, the isomorphism

$$\mathrm{Aut}^o(X \times Y) \cong \mathrm{Aut}^o(X) \times \mathrm{Aut}^o(Y)$$

holds for those proper schemes X and Y such that $\mathcal{O}(X) = \mathcal{O}(Y) = k$, but may fail for arbitrary proper schemes. Indeed, let X be a complete variety having non-zero global vector fields, and let $Y := \mathrm{Spec}\, k[\varepsilon]$ where $\epsilon^2 = 0$; denote by y the closed point of Y. Then we have an exact sequence

$$1 \longrightarrow \Gamma(X, T_X) \longrightarrow \mathrm{Aut}_Y(X \times Y) \longrightarrow \mathrm{Aut}(X) \longrightarrow 1,$$

where the map on the right is obtained by restricting to $X \times \{y\}$. This identifies the vector group $\Gamma(X, T_X)$ to a closed subgroup of $\mathrm{Aut}^o(X \times Y)$, which is not in the image of the natural homomorphism.

Likewise, $\mathrm{Aut}(X \times Y)$ is generally strictly larger than $\mathrm{Aut}(X) \times \mathrm{Aut}(Y)$ (e.g. take $Y = X$ and consider the automorphism $(x, y) \mapsto (y, x)$).

3 Torsors and asssociated fiber bundles

Consider a group scheme G, a G-scheme X, and a G-invariant morphism

$$\pi : X \longrightarrow Y, \tag{2}$$

where Y is a scheme. We say that X is a *G-torsor over Y*, if π is faithfully flat and the morphism

$$\alpha \times p_2 : G \times X \longrightarrow X \times_Y X, \quad (g, x) \longmapsto (g \cdot x, x) \tag{3}$$

is an isomorphism. The latter condition is equivalent to the existence of a faithfully flat morphism $f : Y' \to Y$ such that the pull-back torsor $\pi' : X \times_Y Y' \to Y'$ is trivial. (Since our schemes are assumed to be of finite type, π is quasi-compact and finitely presented; thus, there is no need to distinguish between the fppf and the fpqc topology).

For a G-torsor (2), the morphism π is surjective, and its geometric fiber $X_{\bar{y}}$ is isomorphic to $G_{\bar{y}}$ for any (possibly non-closed) point $y \in Y$. In particular, π is smooth if and only if G is an algebraic group; under that assumption, X is smooth (resp. normal) if and only if so is Y.

Also, note that π is a universal geometric quotient in the sense of [MFK, Section 0], and hence a universal categorical quotient (see [loc. cit., Proposition 0.1]). In particular, $Y(k) = X(k)/G(k)$ and $\mathcal{O}_Y = \pi_*(\mathcal{O}_X)^G$ (the subsheaf of G-invariants in $\pi_*(\mathcal{O}_X)$). Thus, we will also denote Y by X/G.

Remark 3.1. If G is an affine algebraic group, then every G-torsor (2) is *locally isotrivial*, i.e., for any point $y \in Y$ there exist an open subscheme $V \subset Y$ containing y and a finite étale surjective morphism $f : V' \to V$ such that the pull-back torsor $X \times_V V'$ is trivial (this result is due to Grothendieck, see [Ra, Lemme XIV 1.4] for a detailed proof). The local isotriviality of π also holds if G is an algebraic group and Y_{red} is normal, as a consequence of [loc. cit., Théorème XIV 1.2]. In particular, π is locally trivial for the étale topology in both cases.

Yet there exist torsors under algebraic groups that are not locally isotrivial, see [loc. cit., XIII 3.1] (reformulated in more concrete terms in [Br1, Example 6.2]) for an example where Y is a rational nodal curve, and G is an abelian variety having a point of infinite order.

Given a G-torsor (2) and a G-scheme Z, we may view $X \times Z$ as a G-scheme for the diagonal action, and ask if there exist a G-torsor $\varpi : X \times Z \to W$ where W is a scheme, and a morphism $q : W \to Y$ such that the square

$$\begin{array}{ccc} X \times Z & \xrightarrow{p_1} & X \\ {\scriptstyle\varpi}\downarrow & & \downarrow{\scriptstyle\pi} \\ W & \xrightarrow{q} & Y \end{array}$$

is cartesian; here p_1 denotes the first projection. Then q is called the *associated fiber bundle with fiber* Z. The quotient scheme W will be denoted by $X \times^G Z$.

The answer to this question is positive if Z admits an ample G-linearized invertible sheaf (as follows from descent theory; see [SGA1, Proposition 7.8] and also [MFK, Proposition 7.1]). In particular, the answer is positive if Z is affine. Yet the answer is generally negative, even if Z is a smooth variety; see [Bi]. However, associated fiber bundles do exist in the category of algebraic spaces, see [KM, Corollary 1.2].

Of special interest is the case that the fiber is a group scheme G' where G acts through a homomorphism $f : G \to G'$. Then $X' := X \times^G G'$ is a G'-torsor over Y, obtained from X by *extension of the structure group*. If f identifies G with a closed subgroup scheme of G', then X' comes with a G'-morphism to G'/G arising from the projection $X \times G' \to G'$. Conversely, the existence of such a morphism yields a reduction of structure group, in view of the following standard result:

LEMMA 3.2. *Let G be a group scheme, H a subgroup scheme, and X a G-scheme equipped with a G-morphism $f : X \to G/H$. Denote by Z the fiber of f at the base point of G/H, so that Z is an H-scheme.*

Then f is faithfully flat, and the natural map $G \times Z \to X$ factors through a G-isomorphism $G \times^H Z \cong X$.

If $\pi : X \to Y$ is a G-torsor, then the restriction $\pi_{|Z} : Z \to Y$ is an H-torsor.

PROOF. Form and label the cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{f'} & G \\ {\scriptstyle q'}\downarrow & & {\scriptstyle q}\downarrow \\ X & \xrightarrow{f} & G/H \end{array}$$

where q denotes the quotient map. Then X' is a G-scheme and f' is a G-morphism with fiber Z at e_G. It follows readily that the morphism

$$G \times Z \longrightarrow X', \quad (g, z) \longmapsto g \cdot z$$

is an isomorphism, with inverse

$$X' \longrightarrow G \times Z, \quad x' \longmapsto (f'(x'), f'(x')^{-1} \cdot x').$$

This identifies f' with the projection $G \times X \to G$; in particular, f' is faithfully flat. Since q is an H-torsor, f is faithfully flat as well; moreover, q' is an H-torsor. This yields the first assertion.

Next, the G-torsor $\pi : X \to Y$ yields a $G \times H$-torsor

$$F : G \times Z \longrightarrow Y, \quad (g, z) \longmapsto \pi(g \cdot z).$$

Moreover, F is the composite of the projection $G \times Z \to Z$ followed by $\pi_{|Z}$. Thus, $\pi_{|Z}$ is faithfully flat. It remains to show that the natural morphism $H \times Z \to Z \times_Y Z$ is an isomorphism. But this follows by considering the isomorphism (3) and taking the fiber of the morphism $f \times f : X \times_Y X \to G/H \times G/H$ at the base point of $G/H \times G/H$. $\square$

Returning to a G-torsor (2) and a G-scheme Z, we now show that the associated fiber bundle $X \times^G Z$ is a scheme in the case that G is connected and acts transitively on Z. Then $Z \cong G/H$ for some subgroup scheme $H \subset G$, and hence $X \times^G Z \cong X/H$ as algebraic spaces.

THEOREM 3.3. *Let G be a connected group scheme, $\pi : X \to Y$ a G-torsor, and $H \subset G$ a subgroup scheme. Then:*

(i) *π factors uniquely as the composite*

$$X \xrightarrow{p} Z \xrightarrow{q} Y, \tag{4}$$

where Z is a scheme, and p is an H-torsor.

(ii) *If H is a normal subgroup scheme of G, then q is a G/H-torsor.*

PROOF. (i) The uniqueness of the factorization (4) follows from the fact that p is a universal geometric quotient.

Also, the factorization (4) exists after base change under $\pi : X \to Y$: it is just the composite

$$G \times X \xrightarrow{r \times \mathrm{id}_X} G/H \times X \xrightarrow{p_2} X$$

where $r : G \to G/H$ is the quotient map, and p_2 the projection.

Thus, it suffices to show that the algebraic space X/H is representable by a scheme.

We first prove this assertion under the assumption that G *is a (connected) algebraic group*. We begin by reducing to the case that X *and* Y *are normal, quasi-projective varieties*. For this, we adapt the argument of [Ra, pp. 206–207]. We may assume that $X = G \cdot U$, where $U \subset X$ is an open affine subscheme (since X is covered by open G-stable subschemes of that form). Then let $\nu : \tilde{Y} \to Y$ denote the normalization map of Y_{red}. Consider the cartesian square

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & X \\ {\scriptstyle\tilde{\pi}}\downarrow & & \downarrow{\scriptstyle\pi} \\ \tilde{Y} & \xrightarrow{\nu} & Y \end{array}$$

and let $\tilde{U} := U \times_Y \tilde{Y}$. Then $\tilde{\pi}$ is a G-torsor, and hence $\tilde{X}$ is normal. Moreover, $\tilde{X} = G \cdot \tilde{U}$ contains $\tilde{U}$ as an open affine subset. Hence $\tilde{\pi}$ is quasi-projective by [Ra, Théorème VI 2.3]. Therefore, to show that X/H is a scheme, it suffices to check that $\tilde{X}/H$ is a scheme in view of [loc. cit., Lemme XI.3.2]. Thus, we may assume that X is normal and π is

quasi-projective. Then we may further assume that Y is quasi-projective, and hence so is X. Now X is the disjoint union of its irreducible components, and each of them is G-stable; thus, we may assume that X is irreducible. This yields the desired reduction.

Thus, we may assume that there exists an ample invertible sheaf L on X; since X is normal, we may assume that L is G_{aff}-linearized. In view of [Br1, Lemma 3.2], it follows that there exists a G-morphism $X \to G/G_1$, where $G_1 \subset G$ is a subgroup scheme containing G_{aff} and such that G_1/G_{aff} is finite. By Lemma 3.2, this yields a G-isomorphism

$$X \cong G \times^{G_1} X_1$$

where $X_1 \subset X$ is a closed subscheme, stable under G_1. Moreover, the restriction $\pi_1 : X_1 \to Y$ is a G_1-torsor. Since G_1 is affine, so is the morphism π_1 and hence X_1 is quasi-projective.

We now show that π_1 factors as a G_{aff}-torsor $p_1 : X_1 \to X_1/G_{\mathrm{aff}}$, where X_1/G_{aff} is a quasi-projective scheme, followed by a G_1/G_{aff}-torsor $q_1 : X_1/G_{\mathrm{aff}} \to Y$. Indeed, the associated fiber bundle $X_1 \times^{G_1} G_1/G_{\mathrm{aff}}$ is a quasi-projective scheme, since G_1/G_{aff} is affine; we then take for p_1 the composite of the morphism $\mathrm{id}_{X_1} \times e_{G_1} : X_1 \to X_1 \times G_1$ with the natural morphism $X_1 \times G_1 \to X_1 \times^{G_1} G_1/G_{\mathrm{aff}}$. Then p_1 is G_{aff}-invariant and fits into a commutative diagram

$$\begin{array}{ccccc}
X_1 \times G_1 & \longrightarrow & X_1 \times G_1/G_{\mathrm{aff}} & \longrightarrow & X_1 \\
\downarrow & & \downarrow & & \downarrow \\
X_1 & \xrightarrow{p_1} & X_1 \times^{G_1} G_1/G_{\mathrm{aff}} & \xrightarrow{q_1} & X_1/G_1 = Y
\end{array}$$

where the top horizontal arrows are the natural projections, and the vertical arrows are G_1-torsors; thus, p_1 is a G_{aff}-torsor.

Next, note that the smooth, quasi-projective G_{aff}-variety G admits a G_{aff}-linearized ample invertible sheaf. By the preceding step and [MFK, Proposition 7.1], it follows that $G \times^{G_{\mathrm{aff}}} X_1$ is a quasi-projective scheme; it is the total space of a G_1/G_{aff}-torsor over $X = G \times^{G_1} X_1$. Likewise, $G/H \times^{G_{\mathrm{aff}}} X_1$ is a quasi-projective scheme, the total space of a G_1/G_{aff}-torsor over

$$(G/H \times^{G_{\mathrm{aff}}} X_1)/(G_1/G_{\mathrm{aff}}) =: Z$$

It follows that $Z = G/H \times^{G_1} X_1$ fits into a cartesian square

$$\begin{array}{ccc}
G \times X_1 & \xrightarrow{r \times \mathrm{id}_{X_1}} & G/H \times X_1 \\
\downarrow & & \downarrow \\
X & \xrightarrow{p} & Z
\end{array}$$

where the vertical arrows are G_1-torsors; therefore, p is an H-torsor.

Finally, in the *general case*, we may assume that k has characteristic $p > 0$. For any positive integer n, we then have the n-th Frobenius morphism

$$F_G^n : G \longrightarrow G^{(n)}.$$

Its kernel G_n is a finite local subgroup scheme of G. Likewise, we have the n-th Frobenius morphism

$$F_X^n : X \longrightarrow X^{(n)}$$

and $G^{(n)}$ acts on $X^{(n)}$ compatibly with the G-action on X. In particular, F_X^n is invariant under G_n. Since the morphism F_X^n is finite, the sheaf of $\mathcal{O}_{X^{(n)}}$-algebras $((F_X^n)_* \mathcal{O}_X)^{G_n}$ is of finite type. Thus, the scheme

$$X/G_n := \operatorname{Spec}_{X^{(n)}} ((F_X^n)_* \mathcal{O}_X)^{G_n}$$

is of finite type, and F_X^n is the composite of the natural morphisms $X \to X/G_n \to X^{(n)}$. Clearly, the formation of X/G_n commutes with faithfully flat base change; thus, the morphism

$$\pi_n : X \longrightarrow X/G_n$$

is a G_n-torsor, since this holds for the trivial G-torsor $G \times Y \to Y$. As a consequence, π factors through π_n, the G-action on X descends to an action of $G/G_n \cong G^{(n)}$ on X/G_n, and the map $X/G_n \to Y$ is a $G^{(n)}$-torsor. Note that $G^{(n)}$ is reduced, and hence a connected algebraic group, for $n \gg 0$.

Now consider the restriction

$$F_H^n : H \longrightarrow H^{(n)}$$

with kernel $H_n = H \cap G_n$. Then H acts on X/G_n via its quotient $H/H_n \cong H^{(n)} \subset G^{n)}$. By the preceding step, there exists an $H^{(n)}$-torsor $X/G_n \to (X/G_n)/H^{(n)} = X/G_n H$, and hence a $G_n H$-torsor

$$p_n : X \longrightarrow X/G_n H$$

where $X/G_n H$ is a scheme (of finite type). We now set

$$Z := \operatorname{Spec}_{X/G_n H} ((p_n)_* \mathcal{O}_X)^H$$

so that p_n factors through a morphism $p : X \to Z$. Then p is an H-torsor, since the formations of $X/G_n H$ and Z commute with faithfully flat base change, and p is just the natural map $G \times Y \to G/H \times Y$ when π is the trivial torsor over Y. Likewise, the morphism $Z \to X/G_n H$ is finite, and hence the scheme Z is of finite type.

(ii) The composite map

$$G \times X \xrightarrow{\alpha} X \xrightarrow{p} X/H$$

is invariant under the action of $H \times H$ on $G \times X$ via $(h_1, h_2) \cdot (g, x) = (gh_1^{-1}, h_2 \cdot x)$. This yields a morphism $\beta : G/H \times X/H \to X/H$ which is readily seen to be an action. $\square$

Corollary 3.4. *Let again G be a connected group scheme.*

(i) Given two G-torsors $\pi_1 : X_1 \to Y_1$ and $\pi_2 : X_2 \to Y_2$, the associated torsor $X_1 \times X_2 \to X_1 \times^G X_2$ exists.

(ii) Given a homomorphism of group schemes $f : G \to G'$ and a G-torsor $\pi : X \to Y$, the G'-torsor $\pi' : G' \times^G X \to Y$ (obtained by extension of structure groups) exists.

Proof. (i) Apply Theorem 3.3 to the $G \times G$-torsor $X_1 \times X_2 \to Y_1 \times Y_2$ and to the diagonal embedding of G into $G \times G$.

(ii) Denote by $\bar{G}$ the (scheme-theoretic) image of f and by $p : G' \to G'/\bar{G}$ the quotient morphism. Then $p \times \pi : G' \times X \to G'/\bar{G} \times Y$ is a $\bar{G} \times G$-torsor. Moreover, $\bar{G} \times G$ is a connected group scheme, and contains G viewed as the image of the homomorphism $f \times \mathrm{id}$. Applying Theorem 3.3 again yields a G-torsor $G' \times X \to G' \times^G X$. Moreover, the trivial G'-torsor $G' \times X \to X$ descends to a G'-torsor $G' \times^G X \to X/G = Y$. $\square$

Corollary 3.5. *Let G be a connected algebraic group. Then every G-torsor (2) factors uniquely as the composite*

$$X \xrightarrow{p} Z \xrightarrow{q} Y, \tag{5}$$

where Z is a scheme, p is a G_{aff}-torsor, and q is an $A(G)$-torsor. Here p is affine and q is proper.

Moreover, the following conditions are equivalent:

1. *π is quasi-projective.*

2. *q is projective.*

3. *q admits a reduction of structure group to a finite subgroup scheme $F \subset A(G)$.*

4. *π admits a reduction of structure group to an affine subgroup scheme $H \subset G$.*

These conditions hold if X is smooth. In characteristic 0, they imply that q is isotrivial and π is locally isotrivial.

Proof. The existence and uniqueness of the factorization are direct consequences of Theorem 3.3. The assertions on p and q follow by descent theory (see [SGA1, Exposé VIII, Corollaires 4.8, 5.6]).

1⇒2 is a consequence of [Ra, Lemme XIV 1.5 (ii)].

2⇒1 holds since p is affine.

2⇒3 follows from [Br1, Lemma 3.2].

3⇒4. Let $H \subset G$ be the preimage of F. Then $G/H \cong A(G)/F$. By assumption, X/G_{aff} admits an $A(G)$-morphism to $A(G)/F$; this yields a G-morphism $X \to G/H$.

4⇒3. Since $G_{\mathrm{aff}}H$ is affine (as a quotient of the affine group scheme $G_{\mathrm{aff}} \times H$), we may replace H with $G_{\mathrm{aff}}H$. Thus, we may assume that H is the preimage of a finite subgroup scheme $F \subset A(G)$. Then q admits a reduction of structure group to $A(G)/F$.

If X is smooth, then so are Y and Z; in that case, the condition 3 follows from [Ro, Theorem 14] or alternatively from [Ra, Proposition XIII 2.6].

Also, the condition 3 means that $Y \cong A(G) \times^F Z'$ as $A(G)$-torsors over $Z \cong Z'/F$, where Z' is a closed F-stable subscheme of Y. This yields a cartesian square

$$\begin{array}{ccc} A(G) \times Z' & \xrightarrow{p_2} & Z' \\ \big\downarrow & & \big\downarrow \\ Y & \xrightarrow{q} & Z \end{array}$$

where the vertical arrows are F-torsors, and hence étale in characteristic 0. This shows the isotriviality of q. Since p is locally isotrivial, so is π. □

REMARKS 3.6. (i) The equivalent conditions in the preceding result do not generally hold in the setting of normal varieties. Specifically, given an elliptic curve G, there exists a G-torsor $\pi : X \to Y$ where Y is a normal affine surface and X is not quasi-projective; then of course π is not projective (see [Br1, Example 6.4], adapted from [Ra, XIII 3.2]).

(ii) When the condition 4 holds, one may ask whether π admits a reduction of structure group to some affine algebraic subgroup $H \subset G$. The answer is trivially positive in characteristic 0, but negative in characteristic $p > 0$, as shown by the following example.

Choose an integer $n \geq 2$ not divisible by p, and let C denote the curve of equation $y^p = x^n - 1$ in the affine plane $\mathbb{A}^2$, minus all points $(x, 0)$ where x is a n-th root of unity. The group scheme μ_p of p-th roots of unity acts on $\mathbb{A}^2$ via $t \cdot (x, y) = (x, ty)$, and this action leaves C stable. The morphism $\mathbb{A}^2 \to \mathbb{A}^2$, $(x, y) \mapsto (x, y^p)$ restricts to a μ_p-torsor

$$q : C \longrightarrow Y$$

where $Y \subset \mathbb{A}^2$ denotes the curve of equation $y = x^n - 1$ minus all points $(x, 0)$ with $x^n = 1$. Note that Y is smooth, whereas C is singular; both curves are rational, since the equation of C may be rewritten as $x^n = (y + 1)^p$.

Next, let G be an ordinary elliptic curve, so that G contains μ_p, and denote by

$$\pi : X = G \times^{\mu_p} C \longrightarrow Y$$

the G-torsor obtained by extension of structure group (which exists since C is affine). Then X is a smooth surface.

We show that there exists no G-morphism $f : X \to G/H$, where H is an affine algebraic (or equivalently, finite) subgroup of G. Indeed, f would map the rational curve $C \subset X$ and all its translates by G to points of the elliptic curve G/H, and hence f would factor through a G-morphism $G/\mu_p \to G/H$. As a consequence, $\mu_p \subset H$, a contradiction.

(iii) Given a torsor (2) under a group scheme (of finite type) G, there exists a unique factorization

$$X \xrightarrow{p} Z = X/G^o_{\mathrm{red}} \xrightarrow{q} Y \tag{6}$$

where Z is a scheme, p is a torsor under the connected algebraic group G^o_{red}, and q is finite. (Indeed, $Z = X \times^G G/G^o_{\mathrm{red}}$ as in the proof of Theorem 3.3).

4 Automorphism groups of torsors

To any G-torsor $\pi : X \to Y$ as in Section 3, one associates several groups of automorphisms:

- the automorphism group of X as a scheme over Y, denoted by $\mathrm{Aut}_Y(X)$ and called the relative automorphism group,
- the automorphism group of the pair (X, Y), denoted by $\mathrm{Aut}(X, Y)$: it consists of those pairs $(\varphi, \psi) \in \mathrm{Aut}(X) \times \mathrm{Aut}(Y)$ such that the square

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & X \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ Y & \xrightarrow{\psi} & Y \end{array}$$

  commutes,
- the automorphism group of X viewed as a G-scheme, denoted by $\mathrm{Aut}^G(X)$ and called the equivariant automorphism group.

Clearly, the projection $p_2 : \mathrm{Aut}(X) \times \mathrm{Aut}(Y) \to \mathrm{Aut}(Y)$ yields an exact sequence of (abstract) groups

$$1 \longrightarrow \mathrm{Aut}_Y(X) \longrightarrow \mathrm{Aut}(X, Y) \xrightarrow{p_2} \mathrm{Aut}(Y).$$

Also, note that each G-morphism $\varphi : X \to X$ descends to an morphism $\psi : Y \to Y$, since $\pi \circ \varphi : X \to Y$ is G-invariant and π is a categorical quotient. The assignement $\varphi \in \mathrm{Aut}^G(X) \mapsto \psi =: \pi_*(\varphi) \in \mathrm{Aut}(Y)$ yields an identification of $\mathrm{Aut}^G(X)$ with a subgroup of $\mathrm{Aut}(X, Y)$, and an exact sequence of groups

(7) $$1 \longrightarrow \mathrm{Aut}^G_Y(X) \longrightarrow \mathrm{Aut}^G(X) \xrightarrow{\pi_*} \mathrm{Aut}(Y).$$

Moreover, we may view the equivariant automorphisms as those pairs (ϕ, ψ) where $\psi \in \mathrm{Aut}(Y)$, and $\phi : X \to X_\psi$ is a G-morphism. Here X_ψ denotes the G-torsor over Y obtained by pull-back under ψ; note that ϕ is an isomorphism, as a morphism of G-torsors over the same base.

The relative automorphism group is described by the following result, which is certainly well-known but for which we could not locate any appropriate reference:

Lemma 4.1. *Let $\pi : X \to Y$ be a G-torsor. Then the map*

$$\mathrm{Hom}(X, G) \longrightarrow \mathrm{Aut}_Y(X), \quad (f : X \to G) \longmapsto (F : X \to X, \quad x \mapsto f(x) \cdot x)$$

is an isomorphism of groups, which restricts to an isomorphism

(8) $$\mathrm{Hom}^G(X, G) \cong \mathrm{Aut}^G_Y(X).$$

Here $\mathrm{Hom}^G(X, G) \subset \mathrm{Hom}(X, G)$ denotes the subset of morphisms that are equivariant for the given G-action on X, and the $G-$action on itself by conjugation.

If G is commutative, then $\mathrm{Aut}^G_Y(X) \cong \mathrm{Hom}(Y, G)$.

Proof. Let $u \in \mathrm{Aut}_Y(X)$. Then $u \times \mathrm{id}_X$ is an automorphism of $X \times_Y X$ over X. In view of the isomorphism (3), $u \times \mathrm{id}_X$ yields an automorphism of $G \times X$ over X, thus of the form $(g, x) \mapsto (F(g, x), x)$ for a unique $F \in \mathrm{Hom}(G \times X, G)$. In other words, $u(g \cdot x) = F(g, x) \cdot x$. Thus, $u(x) = f(x) \cdot x$, where $f := F(e_G, -) \in \mathrm{Hom}(X, G)$. This yields the claimed isomorphism $\mathrm{Hom}(X, G) \cong \mathrm{Aut}_Y(X)$, equivariant for the action of G on $\mathrm{Hom}(X, G)$ via $(g \cdot f)(x) = gf(g^{-1} \cdot x)g^{-1}$ and on $\mathrm{Aut}_Y(X)$ by conjugation. Taking invariants, we obtain the isomorphism (8).

If G is commutative, then $\mathrm{Hom}^G(X, G)$ consists of the G-invariant morphisms $X \to G$; these are identified with the morphisms $Y = X/G \to G$. $\square$

The preceding considerations adapt to group functors of automorphisms, that associate to any scheme S the groups $\mathrm{Aut}_{Y \times S}(X \times S)$, $\mathrm{Aut}_S(X \times S, Y \times S)$ and their equivariant analogues. We will denote these functors by $Aut_Y(X)$, $Aut(X, Y)$, $Aut^G(X)$ and $Aut^G_Y(X)$. The exact sequence (7) readily yields an exact sequence of group functors

(9) $$1 \longrightarrow Aut^G_Y(X) \longrightarrow Aut^G(X) \xrightarrow{\pi_*} Aut(Y).$$

Also, by Lemma 4.1, we have a functorial isomorphism

$$\mathrm{Aut}_{Y \times S}(X \times S) \cong \mathrm{Hom}(X \times S, G).$$

In other words, $Aut_Y(X)$ is isomorphic to the group functor

$$Hom(X,G) : S \longmapsto \mathrm{Hom}(X \times S, G).$$

As a consequence, $Aut_Y^G(X)$ is isomorphic to $Hom^G(X,G) : S \mapsto \mathrm{Hom}^G(X \times S, G)$. This readily yields isomorphisms

$$\mathrm{Lie}\,\mathrm{Aut}_Y(X) \cong \mathrm{Hom}(X, \mathrm{Lie}(G)) \cong \mathcal{O}(X) \otimes \mathrm{Lie}(G),$$

$$\mathrm{Lie}\,\mathrm{Aut}_Y^G(X) \cong \mathrm{Hom}^G(X, \mathrm{Lie}(G)) \cong (\mathcal{O}(X) \otimes \mathrm{Lie}(G))^G.$$

We now obtain a finiteness result for $Aut^G(X)$, analogous to a theorem of Morimoto (see [Mo, Théorème, p. 158]):

THEOREM 4.2. *Consider a G-torsor $\pi : X \to Y$ where G is a group scheme, X a scheme, and Y a proper scheme. Then the functor $Aut^G(X)$ is represented by a group scheme, locally of finite type, with Lie algebra $\Gamma(X, T_X)^G$.*

PROOF. The assertion on the Lie algebra follows from the G-isomorphism (1).

To show the representability assertion, we first reduce to the case that G is a connected affine algebraic group. Let G_{aff} denote the largest closed normal affine subgroup of G, or equivalently of G^o_{red}. Then $Aut^G(X)$ is a closed subfunctor of $Aut^{G_{\mathrm{aff}}}(X)$. Moreover, the factorizations (5) and (6) yield a factorization of π as

$$X \xrightarrow{\ p\ } X/G_{\mathrm{aff}} \xrightarrow{\ q\ } X/G^o_{\mathrm{red}} \xrightarrow{\ r\ } Y$$

where p is a torsor under G_{aff}, q a torsor under $G^o_{\mathrm{red}}/G_{\mathrm{aff}}$, and r is a finite morphism. Since q and r are proper, X/G_{aff} is proper as well. This yields the desired reduction.

Next, we may embed G as a closed subgroup of $\mathrm{GL}(V)$ for some finite-dimensional vector space V. Let Z denote the closure of G in the projective completion of $\mathrm{End}(V)$. Then Z is a projective variety equipped with an action of $G \times G$ (arising from the $G \times G$-action on $\mathrm{End}(V)$ via left and right multiplication) and with an ample $G \times G$-linearized invertible sheaf. By construction, G (viewed as a $G \times G$-variety via left and right multiplication) is the open dense $G \times G$-orbit in Z.

As seen in Section 3, the associated fiber bundle $X \times^G Z$ (for the left G-action on Z) exists; it is equipped with a G-action arising from the right G-action on Z. Moreover, $X \times^G Z$ contains $X \times^G G \cong X$ as a dense open G-stable subscheme. Also, recall the cartesian square

$$\begin{array}{ccc} X \times Z & \xrightarrow{\ p\ } & X \\ {\scriptstyle \varpi}\downarrow & & \downarrow{\scriptstyle \pi} \\ X \times^G Z & \xrightarrow{\ q\ } & Y. \end{array}$$

Since Z is complete and π is faithfully flat, it follows that q is proper, and hence so is $X \times^G Z$.

Now let S be a scheme, and $\varphi \in \mathrm{Aut}_S^G(X \times S)$. Then φ yields an S-automorphism

$$\phi : X \times Z \times S \longrightarrow X \times Z \times S, \quad (x, z, s) \longmapsto (\varphi(x, s), z, s).$$

Consider the action of $G \times G$ on $X \times Z \times S$ given by

$$(g_1, g_2) \cdot (x, z, s) = (g_1 \cdot x, (g_1, g_2) \cdot z, s).$$

Then ϕ is $G \times G$-equivariant, and hence yields an automorphism $\Phi \in \mathrm{Aut}_S^G(X \times^G Z \times S)$ which stabilizes $X \times^G (Z \backslash G) \times S$. Moreover, the assignement $\varphi \mapsto \Phi$ identifies $\mathrm{Aut}_S^G(X \times S)$ with the stabilizer of $X \times^G (Z \setminus G) \times S$ in $\mathrm{Aut}_S^G(X \times^G Z \times S)$. Thereby, $Aut^G(X)$ is identified with a closed subfunctor of $Aut(X \times^G Z)$; the latter is represented by a group scheme of finite type, since $X \times^G Z$ is proper. $\square$

For simplicity, we denote by $\mathrm{Aut}^G(X)$ the group scheme defined in the preceding theorem. Since $Aut_Y^G(X)$ is a closed subfunctor of $Aut^G(X)$, it is also represented by a group scheme (locally of finite type) that we denote likewise by $\mathrm{Aut}_Y^G(X)$. Further properties of this relative automorphism group scheme are gathered in the following:

Proposition 4.3. *Let $\pi : X \to Y$ be a torsor under a connected algebraic group G, where Y is a proper scheme. Then the factorization $X \xrightarrow{p} Z = X/G_{\mathrm{aff}} \xrightarrow{q} Y$ (obtained in Corollary 3.5) yields an exact sequence of group schemes*

$$1 \longrightarrow \mathrm{Aut}_Z^{G_{\mathrm{aff}}}(X) \longrightarrow \mathrm{Aut}_Y^G(X) \xrightarrow{p_*} \mathrm{Aut}_Y^{A(G)}(Z). \tag{10}$$

Moreover, $\mathrm{Aut}_Z^{G_{\mathrm{aff}}}(X)$ is affine of finite type,

If Y is a (complete) variety, then the neutral component of $\mathrm{Aut}_Y^{A(G)}(Z)$ is just $A(G)$; it is contained in the image of p_*.

Proof. We first show that $\mathrm{Aut}_Z^{G_{\mathrm{aff}}}(X)$ is affine of finite type. By Lemma 4.1, we have

$$Aut_Z^{G_{\mathrm{aff}}}(X) \cong Hom^{G_{\mathrm{aff}}}(X, G_{\mathrm{aff}}).$$

Moreover, there exists a closed G_{aff}-equivariant immersion of G_{aff} into an affine space V where G_{aff} acts via a representation. Thus, $Aut_Z^{G_{\mathrm{aff}}}(X)$ is a closed subfunctor of $Hom^{G_{\mathrm{aff}}}(X, V)$. But the latter is represented by an affine space (of finite dimension), namely, the space of global sections of the associated vector bundle $X \times^{G_{\mathrm{aff}}} V$ over the proper scheme $X/G_{\mathrm{aff}} = Z$. This completes the proof.

Next, we obtain (10). We start with the exact sequence (9) for the G_{aff}-torsor p, which translates into an exact sequence of group schemes

$$1 \longrightarrow \mathrm{Aut}_Z^{G_{\mathrm{aff}}}(X) \longrightarrow \mathrm{Aut}^{G_{\mathrm{aff}}}(X) \xrightarrow{p_*} \mathrm{Aut}(Z).$$

Taking G-invariants yields the exact sequence of group schemes

$$1 \longrightarrow \mathrm{Aut}_Z^G(X) \longrightarrow \mathrm{Aut}_Y^G(X) \xrightarrow{p_*} \mathrm{Aut}(Z).$$

But G acts on the affine scheme $\mathrm{Aut}_Z^{G_{\mathrm{aff}}}(X)$ through its quotient $G/G_{\mathrm{aff}} = A(G)$, an abelian variety. So this G-action must be trivial, that is, $\mathrm{Aut}_Z^G(X) = \mathrm{Aut}_Z^{G_{\mathrm{aff}}}(X)$.

We now show that $A(G) = \mathrm{Aut}_Y^{A(G),o}(Z)$ if Y (or equivalently Z) is a variety. Since $A(G)$ is commutative, we have a homorphism $f : A(G) \to \mathrm{Aut}_Y^{A(G)}(Z)$. The induced homomorphism of Lie algebras is the natural map

$$\mathrm{Lie}\, A(G) \longrightarrow \mathrm{Lie}\, \mathrm{Aut}_Y^{A(G)}(Z) = (\mathcal{O}(Z) \otimes \mathrm{Lie}\, A(G))^{A(G)}$$

which is an isomorphism since $\mathcal{O}(Z) = k$. This yields our assertion.

Finally, we show that $A(G)$ is contained in the image of p_*. Indeed, the neutral component of the center of G is identified with a subgroup of $\mathrm{Aut}_Y^G(X)$, and is mapped onto $A(G)$ under the quotient homomorphism $G \to G/G_{\mathrm{aff}}$ (as follows from [Ro, Corollary 5, p. 440]). $\square$

Observe that the exact sequence (10) yields an analogue for torsors of Chevalley's structure theorem; it gives back that theorem when applied to the trivial torsor G.

5 Lifting automorphisms for abelian torsors

We begin by determining the relative equivariant automorphism groups of torsors under abelian varieties:

Proposition 5.1. *Let G be an abelian variety and $\pi : X \to Y$ a G-torsor, where X and Y are complete varieties. Then the group scheme $\mathrm{Aut}_Y^G(X)$ is isomorphic to* $\mathrm{Hom}_{\mathrm{gp}}(A(Y), G) \times G$. *Here $A(Y)$ denotes the Albanese variety of Y, and* $\mathrm{Hom}_{\mathrm{gp}}(A(Y), G)$ *denotes the space of homomorphisms of algebraic groups $A(Y) \to G$; this is a free abelian group of finite rank, viewed as a constant group scheme.*

Proof. By Lemma 4.1, we have a functorial isomorphism

$$\mathrm{Aut}_{Y \times S}^G(X \times S) \cong \mathrm{Hom}(Y \times S, G).$$

Choose a point $y_0 \in Y$. For any $f \in \mathrm{Hom}(Y \times S, G)$, consider the morphism

$$\varphi : Y \times S \longrightarrow G, \quad (y, s) \longmapsto f(y, s) - f(y_0, s)$$

where the group law of the abelian variety G is denoted additively. We claim that φ factors through the projection $Y \times S \to Y$. For this, we may replace k with a larger field,

and assume that S has a k-rational point s_0; we may also assume that S is connected. Then the morphism

$$\psi : Y \times S \longrightarrow G, \quad (y, s) \longmapsto f(y, s) - f(y, s_0)$$

maps $Y \times \{s_0\}$ to a point. By a scheme-theoretic version of the rigidity lemma (see [SS, Theorem 1.7]), it follows that ψ factors through the projection $Y \times S \to S$. Thus, $f(y, s) - f(y, s_0) = f(y_0, s) - f(y_0, s_0)$ which shows the claim.

By that claim, we may write

$$f(y, s) = \varphi(y) + \psi(s)$$

where $\varphi : Y \to G$ and $\psi : S \to G$ are morphisms such that $\varphi(y_0) = 0$. Now let $a : Y \to A(Y)$ be the Albanese morphism, normalized so that $a(y_0) = 0$. Then φ factors through a unique homorphism $\Phi : A(Y) \to A$, and $f = (\Phi \circ a) + \psi$ where Φ is an S-point of $\mathrm{Hom}_{\mathrm{gp}}(A(Y), G)$, and ψ an S-point of G. □

Next, we obtain a preliminary result which again is certainly well-known, but for which we could not locate any reference:

Lemma 5.2. *Assume that k has characteristic 0. Let $\pi : Z \to Y$ be a finite étale morphism, where Y and Z are complete varieties.*

Then the natural homomorphism $\pi_* : \mathrm{Aut}(Z, Y) \to \mathrm{Aut}(Y)$ restricts to an isogeny $\mathrm{Aut}^o(Z, Y) \to \mathrm{Aut}^o(Y)$ on neutral components.

If π is a Galois cover with group F (that is, an F-torsor), then $\mathrm{Aut}^o(Z, Y)$ is the neutral component of $\mathrm{Aut}^F(Y)$.

Proof. We set for simplicity $H := \mathrm{Aut}^o(Y)$; this is a connected algebraic group in view of the characteristic-0 assumption. For any $h \in H(k)$, denote by Z_h the étale cover of Y obtained from Z by pull-back under h. Then the covers Z_h, $h \in H(k)$, are all isomorphic by [SGA1, Exposé X, Corollaire 1.9]. Thus, every $h \in H(k)$ lifts to some $\tilde{h} \in \mathrm{Aut}(Z)(k)$. In other words, the image of the projection $\pi_* : \mathrm{Aut}(Z, Y) \to \mathrm{Aut}(Y)$ contains H. It follows that π_* restricts to a surjective homomorphism $\mathrm{Aut}^o(Z, Y) \to \mathrm{Aut}^o(Y)$; its kernel is finite by Galois theory.

If π is an F-torsor, then π_* has kernel F, by Galois theory again. In particular, $\mathrm{Aut}(Z, Y)$ normalizes F. The action of the neutral component $\mathrm{Aut}^o(Z, Y)$ by conjugation on the finite group F must be trivial; this yields the second assertion. □

Remark 5.3. In particular, with the notation and assumptions of the preceding lemma, all elements of $\mathrm{Aut}^o(Y)$ lift to automorphisms of Z. But this does not generally hold for elements of $\mathrm{Aut}(Y)$. For a very simple example, take $k = \mathbb{C}$, Z the elliptic curve $\mathbb{C}/2\mathbb{Z}+i\mathbb{Z}$, Y the elliptic curve $\mathbb{C}/\mathbb{Z} + i\mathbb{Z}$, and π the natural morphism. Then the multiplication by i defines an automorphism of Y which admits no lift under the double cover π.

We now come to the main result of this section:

THEOREM 5.4. *Let G be an abelian variety and $\pi : X \to Y$ a G-torsor, where X and Y are complete varieties. Then G centralizes $\mathrm{Aut}^o(X)$; equivalently, $\mathrm{Aut}^o(X) = \mathrm{Aut}^{G,o}(X)$. Moreover, there exists a closed subgroup $H \subset \mathrm{Aut}^o(X)$ such that $\mathrm{Aut}^o(X) = GH$ and $G \cap H$ is finite.*

If k has characteristic 0 and X (or equivalently Y) is smooth, then the homomorphism $\pi_* : \mathrm{Aut}^G(X) \longrightarrow \mathrm{Aut}(Y)$ restricts to an isogeny $\pi_{*|H} : H \to \mathrm{Aut}^o(Y)$ for any quasi-complement H as above.

PROOF. The assertion that G is central in $\mathrm{Aut}^o(X)$ and admits a quasi-complement follows from [Ro, Corollary, p. 434].

By Proposition 4.3 or alternatively Proposition 5.1, G is the neutral component of the kernel of π_*. Thus, the kernel of $\pi_{*|H}$ is finite.

It remains to show that $\pi_{*|H}$ is surjective when k has characteristic 0 and X is smooth. By Lemma 3.2 and Corollary 3.5, we have a G-isomorphism

$$X \cong G \times^F Z \tag{11}$$

for some finite subgroup $F \subset G$ and some closed F-stable subscheme $Z \subset X$ such that $\pi : Z \to Y$ is an F-torsor. Thus, Z is smooth and complete. Replacing Z with a component, and F with the stabiliser of that component, we may assume that F is a variety. Then by Lemma 5.2, the natural homomorphism $\mathrm{Aut}^{F,o}(Z) \to \mathrm{Aut}^o(Y)$ is surjective.

We now claim that $\mathrm{Aut}^F(Z)$ may be identified with a closed subgroup of $\mathrm{Aut}^G(X)$. Indeed, as in the proof of Theorem 4.2, any $\varphi \in \mathrm{Aut}^F(X)$ yields a morphism

$$\phi : G \times Z \longrightarrow G \times Z, \quad (g, z) \longmapsto (g, \varphi(z)).$$

This is a $G \times F$-automorphism of $X \times Z$, and hence descends to a G-automorphism Φ of X. The assignement $\varphi \mapsto \Phi$ yields the desired identification. This proves the claim and, in turn, the surjectivity of $\pi_{*|H}$. $\square$

REMARKS 5.5. (i) With the notation and assumptions of the preceding theorem, the surjectivity of $\pi_{*|H}$ also holds when X (or equivalently Y) is normal. Choose indeed an $\mathrm{Aut}^o(Y)$-equivariant desingularization

$$f : Y' \longrightarrow Y,$$

that is, f is proper and birational, and the action of $\mathrm{Aut}^o(Y)$ on Y lifts to an action on Y' such that f is equivariant (see [EV] for the existence of such desingularizations). Since Y is normal, we have $f_*(\mathcal{O}_{Y'}) = \mathcal{O}_Y$. In view of Proposition 2.1, this yields a homomorphism

$$f_* : \mathrm{Aut}^o(Y') \longrightarrow \mathrm{Aut}^o(Y)$$

which is injective (on closed points) as f is birational, and surjective by construction. Thus, f_* is an isomorphism. Likewise, the natural map $\mathrm{Aut}^o(X') \to \mathrm{Aut}^o(X)$ is an isomorphism, where $X' := X \times_Y Y'$ is the total space of the pull-back torsor $\pi' : X' \to Y'$. Now the desired surjectivity follows from Theorem 5.4.

We do not know whether $\pi_{*|H}$ is surjective for arbitrary (complete) varieties X, Y. Also, we do not know whether the characteristic-0 assumption can be omitted.

(ii) The preceding theorem may be reformulated in terms of vector fields only: let X, Y be smooth complete varieties over an algebraically closed field of characteristic 0, and $\pi : X \to Y$ a smooth morphism such that the relative tangent bundle T_π is trivial. Then every global vector on Y lifts to a global vector field on X.

Consider indeed the Stein factorization of π,

$$X \xrightarrow{\pi'} X' \xrightarrow{p} Y.$$

Then one easily checks that p is étale; thus, X' is smooth and π' is smooth with trivial relative tangent bundle. Also, every global vector field on Y lifts to a global vector field on X', as follows e.g. from Lemma 5.2. Thus, we may replace π with π', and hence assume that the fibers of π are connected. Then these fibers are just the orbits of $G := \mathrm{Aut}^o_Y(X)$, an abelian variety. Moreover, for F and Z as in (11), the restriction $\pi_{|Z}$ is smooth, since so is π. Thus, $\pi_{|Z}$ is an F-torsor. So the claim follows again from Theorem 5.4.

Finally, using the factorization (6) and combining Lemma 5.2 and Theorem 5.4, we obtain the following:

Corollary 5.6. *Let G be a proper algebraic group and $\pi : X \to Y$ a G-torsor, where Y is a complete variety over an algebraically closed field of characteristic* 0. *Then there exists a closed connected subgroup $H \subset \mathrm{Aut}^G(X)$ which is isogenous to $\mathrm{Aut}^o(Y)$ via $\pi_* : \mathrm{Aut}^G(X) \to \mathrm{Aut}(Y)$.*

Here the assumption that G is proper cannot be omitted. For example, let Y be an abelian variety, so that $\mathrm{Aut}^o(Y)$ is the group of translations. Let also G be the multiplicative group $\mathbb{G}_m$, so that G-torsors $\pi : X \to Y$ correspond bijectively to invertible sheaves $\mathcal{L}$ on Y. Then $\mathrm{Aut}^o(Y)$ lifts to an isomorphic (resp. isogenous) subgroup of $\mathrm{Aut}^G(X)$ if and only if $\mathcal{L}$ is trivial (resp. of finite order). Also, the image of π_* contains $\mathrm{Aut}^o(Y)$ if and only if $\mathcal{L}$ is algebraically trivial (see [Mu] for these results).

This is the starting point of the theory of homogeneous bundles over abelian varieties, developed in [Br2].

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